Roots of Unity

1. Definition:

For any positive integer n, the n^{th} roots of unity are the complex solutions to $x^n = 1$.

2. Examples:

E.g. 1 Find the 3^{rd} roots of unity.

We can use the formula:

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right) \text{ for } k = 0, 1, \ldots n-1$$

to solve this.

1. Find r

$$r = \sqrt{a^2 + b^2}$$
$$= \sqrt{1^2 + 0^2}$$
$$= 1$$
$$r^{\frac{1}{3}} = 1$$

2. Find θ

$$\theta = \arctan\left(\frac{b}{a}\right)$$
$$= \arctan\left(\frac{0}{1}\right)$$
$$= \arctan(0)$$
$$= 0$$

3. Plug it into the equation and solve

$$z^{\frac{1}{3}} = 1\left(\cos\left(\frac{0}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{0}{3} + \frac{2k\pi}{3}\right)\right)$$
$$= \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$$

When $k = 0$,

$$z^{\frac{1}{3}} = \cos(0) + i\sin(0)$$
$$= 1$$

when $k = 1$,

$z^{\frac{1}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$= -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$= \dfrac{i\sqrt{3} - 1}{2}$

when $k = 2$,

$z^{\frac{1}{3}} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$

$= -\frac{1}{2} + -\frac{i\sqrt{3}}{2}$

$= \dfrac{-1 - i\sqrt{3}}{2}$

To check if your answers are right, raise them to the power of n and see if you get 1.

I.e. If $z^{\frac{1}{n}} = r$ is your solution, then r^n should equal 1. This is because of the definition of roots of unity.

The solutions for $z^{\frac{1}{3}}$ are 1, $\dfrac{i\sqrt{3}-1}{2}$, $\dfrac{-i\sqrt{3}-1}{2}$

1. $(1)^3 = 1 \therefore 1$ is correct.

2. $\left(\dfrac{i\sqrt{3}-1}{2}\right)^3$

$= \frac{1}{8}\left(i\sqrt{3} - 1\right)^3$

$= \frac{1}{8}\left((i\sqrt{3})^3 - 3(i\sqrt{3})^2 + 3(i\sqrt{3}) - 1\right)$

$= \frac{1}{8}\left(-i3\sqrt{3} + 9 + i3\sqrt{3} - 1\right)$

$= \dfrac{8}{8}$

$= 1 \therefore \left(\dfrac{i\sqrt{3}-1}{2}\right)$ is correct

3. $\left(\frac{-i\sqrt{3}-1}{2}\right)^3$

$= \frac{-1}{8}(i\sqrt{3}+1)^3$

$= \frac{-1}{8}\left((i\sqrt{3})^3 + 3(i\sqrt{3})^2 + 3(i\sqrt{3}) + 1\right)$

$= \frac{-1}{8}\left(-i3\sqrt{3} - 9 + i3\sqrt{3} + 1\right)$

$= \frac{8}{8}$

$= 1$

$\therefore \left(\frac{-i\sqrt{3}-1}{2}\right)^3$ is correct

E.g. 2 Find the 4^{th} roots of unity.

Step 1. Find r

$r = \sqrt{a^2+b^2}$
$= \sqrt{1^2+0^2}$
$= 1$
$r^{\frac{1}{4}} = 1$

Step 2. Find θ

$\theta = \arctan\left(\frac{b}{a}\right)$
$= \arctan\left(\frac{0}{1}\right)$
$= 0$

Step 3. Plug it into the equation.

$z^{\frac{1}{4}} = \cos\left(\frac{2k\pi}{4}\right) + i\sin\left(\frac{2k\pi}{4}\right)$
$= \cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right)$

When $k=0$,
$\cos(0) + i\sin(0) = 1$

When $k=1$
$\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$

When $k=2$
$\cos(\frac{2\pi}{2}) + i\sin(\frac{2\pi}{2}) = -1$

When $k=3$
$\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}) = -i$

To check:

1. $(1)^4 = 1$
2. $(i)^4 = 1$
3. $(-1)^4 = 1$
4. $(-i)^4 = 1$

$\therefore$ The answers are right.

3. Primitive Roots of Unity

1. Def:

Let n be a positive integer. A primitive n^{th} root of unity is a n^{th} root of unity that is not a k^{th} root of unity for any k less than n.

I.e. If r is a n^{th} root of unity, then r is a primitive n^{th} root of unity iff $r^n = 1$ and $r^k \neq 1$ for all k s.t. $0 < k < n$.

2. Examples:

1. From our 3^{rd} roots of unity, we got 1, $\frac{i\sqrt{3}-1}{2}$ and $\frac{-i\sqrt{3}-1}{2}$ as our answers.

   We know that $(1)^1 = 1$, and $1<3$, so 1 is not a primitive root of unity.
   However, the other 2 are.

2. From our 4^{th} roots of unity example, we got $1, -1, i$ and $-i$ as our answers.
   We know that $(1)^1 = 1$ and $1<4$, so 1 is not a primitive root of unity. Furthermore, $(-1)^2 = 1$ and $2<4$, so -1 is not a primitive root of unity either. The other 2 are primitive roots of unity.

3. How to find the number of primitive roots:

If you got n roots of unity, find the number of co-primes with n from 1 to n-1. That number is the number of primitive roots.

E.g. 1 We found 2 primitive roots for our 3^{rd} root of unity example. That makes sense because there are 2 co-primes with 3 from 1 to 2.

E.g. 2 We found 2 primitive roots for our 4^{th} root of unity example. From 1 to 3, there are 2 co-primes with 4; 1 and 3. Therefore, it makes sense that we found 2 primitive roots.

Note: This process uses Euler's totient function.

Definitions:

1. Co-prime: 2 integers, a and b, are co-prime if $\gcd(a,b)=1$. I.e. The only positive integer that divides both of them is 1.

   E.g. 3 and 5 are coprimes.
   6 and 1024 are not coprimes.

2. Euler's Totient Function: This counts the number of integers that are co-prime with n, from 1 to n-1. It is denoted as $\phi(n)$.

   E.g. $\phi 15 = 8$ because from 1 to 14, only $\{1,2,4,7,11,13,14\}$ are co-primes with 15.

Powers of i:

$i^1 = i$ $\quad$ $i^5 = i$
$i^2 = -1$ $\quad$ $i^6 = -1$
$i^3 = -i$ $\quad$ $i^7 = -i$
$i^4 = 1$ $\quad$ $i^8 = 1$

This pattern repeats.

In general,

$i^{4n+1} = i, \quad n \in N$
$i^{4n+2} = -1, \quad n \in N$
$i^{4n+3} = -i, \quad n \in N$
$i^{4n+4} = 1, \quad n \in N$